MATHEMATICS ADMISSIONS TEST

For candidates applying for Mathematics, Computer Science or one of their joint degrees at OXFORD UNIVERSITY and/or IMPERIAL COLLEGE LONDON

Wednesday 6 November 2013

Time Allowed: 2½ hours

Please complete the following details in BLOCK CAPITALS

Surname	
Other names	

Candidate Number	**M**						

I am applying to

Y/N	Oxford University	For the degree course:	
Y/N	Imperial College London	For the degree course:	

This paper contains 7 questions of which you should attempt 5. There are directions throughout the paper as to which questions are appropriate for your course.

A: Oxford Applicants: if you are applying to Oxford for the degree course:

- Mathematics *or* Mathematics & Philosophy *or* Mathematics & Statistics, you should attempt Questions **1,2,3,4,5.**
- Mathematics & Computer Science, you should attempt Questions **1,2,3,5,6.**
- Computer Science *or* Computer Science & Philosophy, you should attempt **1,2,5,6,7.**

Directions under A take priority over any directions in B which are relevant to you.

B: Imperial Applicants: if you are applying to Imperial College for any of the Mathematics courses**:** Mathematics, Mathematics (Pure Mathematics), Mathematics with a Year in Europe, Mathematics with Applied Mathematics/Mathematical Physics, Mathematics with Mathematical Computation, Mathematics with Statistics, Mathematics with Statistics for Finance, Mathematics Optimisation and Statistics, you should attempt Questions **1,2,3,4,5.**

Further credit cannot be obtained by attempting extra questions.

Question 1 is a multiple choice question with ten parts. Marks are given solely for correct answers but any rough working should be shown in the space between parts. Answer Question 1 on the grid on Page 2. Each part is worth 4 marks.

Answers to questions 2-7 should be written in the space provided, continuing on to the blank pages at the end of this booklet if necessary. Each of Questions 2-7 is worth 15 marks.

Do not write on this page

MATHEMATICS ADMISSIONS TEST

Wednesday 6 November 2013

Time Allowed: 2½ hours

Please complete these details below in block capitals.

Surname	
Other names	
Centre Number	
Candidate Number	**M**
UCAS Number (if known)	– –
Date of Birth	d d – m m – y y

Please tick the appropriate box:

☐ I have attempted Questions **1,2,3,4,5**

☐ I have attempted Questions **1,2,3,5,6**

☐ I have attempted Questions **1,2,5,6,7**

Admissions Testing Service

Administered on behalf of the University of Oxford and Imperial College London by The Admissions Testing Service

FOR OFFICE USE ONLY

Q1	Q2	Q3	Q4	Q5	Q6	Q7

1. For **ALL APPLICANTS**.

For each part of the question on pages 3–7 you will be given four possible answers, just one of which is correct. Indicate for each part **A**–**J** which answer (a), (b), (c), or (d) you think is correct with a tick (✓) in the corresponding column in the table below. *Please show any rough working in the space provided between the parts.*

	(a)	(b)	(c)	(d)
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

A. For what values of the real number a does the quadratic equation

$$x^2 + ax + a = 1$$

have distinct real roots?

(a) $a \neq 2$; (b) $a > 2$; (c) $a = 2$; (d) all values of a.

B. The graph of $y = \sin x$ is reflected first in the line $x = \pi$ and then in the line $y = 2$. The resulting graph has equation

(a) $y = \cos x$; (b) $y = 2 + \sin x$; (c) $y = 4 + \sin x$; (d) $y = 2 - \cos x$.

Turn over

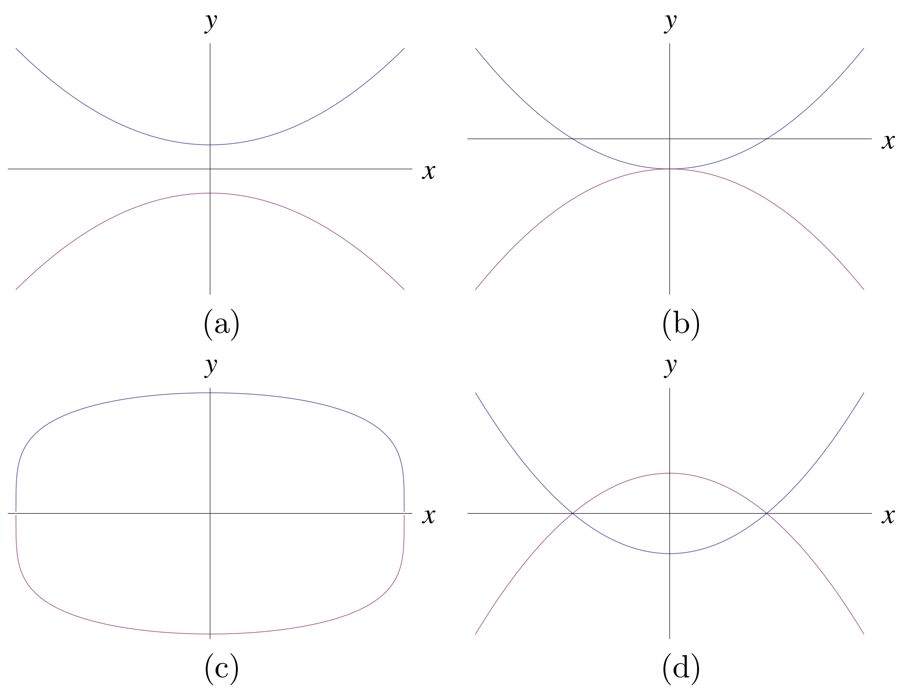
C. The functions f, g and h are related by

$$f'(x) = g(x+1), \qquad g'(x) = h(x-1).$$

It follows that $f''(2x)$ equals

(a) $h(2x+1)$; (b) $2h'(2x)$; (c) $h(2x)$; (d) $4h(2x)$.

D. Which of the following sketches is a graph of $x^4 - y^2 = 2y + 1$?



E. The expression

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[(2x-1)^4(1-x)^5\right]+\frac{\mathrm{d}}{\mathrm{d}x}\left[(2x+1)^4\left(3x^2-2\right)^2\right]$$

is a polynomial of degree

(a) 9; (b) 8; (c) 7; (d) less than 7.

F. Three *positive* numbers a, b, c satisfy

$$\log_b a = 2, \qquad \log_b(c-3) = 3, \qquad \log_a(c+5) = 2.$$

This information

(a) specifies a uniquely.
(b) is satisfied by two values of a.
(c) is satisfied by infinitely many values of a.
(d) is contradictory.

Turn over

G. Let $n \geqslant 2$ be an integer and $p_n(x)$ be the polynomial

$$p_n(x) = (x-1) + (x-2) + \cdots + (x-n).$$

What is the remainder when $p_n(x)$ is divided by $p_{n-1}(x)$?

(a) $\dfrac{n}{2}$; (b) $\dfrac{n+1}{2}$; (c) $\dfrac{n^2+n}{2}$; (d) $\dfrac{-n}{2}$.

H. The area bounded by the graphs

$$y = \sqrt{2-x^2} \qquad \text{and} \qquad x + \left(\sqrt{2}-1\right) y = \sqrt{2}$$

equals

(a) $\dfrac{\sin\sqrt{2}}{\sqrt{2}}$; (b) $\dfrac{\pi}{4} - \dfrac{1}{\sqrt{2}}$; (c) $\dfrac{\pi}{2\sqrt{2}}$; (d) $\dfrac{\pi^2}{6}$.

I. The function $F(k)$ is defined for positive integers by $F(1) = 1$, $F(2) = 1$, $F(3) = -1$ and by the identities

$$F(2k) = F(k), \qquad F(2k+1) = F(k)$$

for $k \geqslant 2$. The sum

$$F(1) + F(2) + F(3) + \cdots + F(100)$$

equals

(a) -15; (b) 28; (c) 64; (d) 81.

J. For a real number x we denote by $[x]$ the largest integer less than or equal to x.

Let n be a natural number. The integral

$$\int_0^n [2^x]\, \mathrm{d}x$$

equals

(a) $\log_2((2^n - 1)!)$; (b) $n2^n - \log_2((2^n)!)$; (c) $n2^n$; (d) $\log_2((2^n)!)$,

where $k! = 1 \times 2 \times 3 \times \cdots \times k$ for a positive integer k.

Turn over

2. For **ALL APPLICANTS.**

(i) Let $k \neq \pm 1$. The function $f(t)$ satisfies the identity

$$f(t) - kf(1-t) = t$$

for all values of t. By replacing t with $1-t$, determine $f(t)$.

(ii) Consider the new identity

$$f(t) - f(1-t) = g(t). \qquad (*)$$

(a) Show that no function $f(t)$ satisfies $(*)$ when $g(t) = t$.

(b) What condition must the function $g(t)$ satisfy for there to be a solution $f(t)$ to $(*)$?

(c) Find a solution $f(t)$ to $(*)$ when $g(t) = (2t-1)^3$.

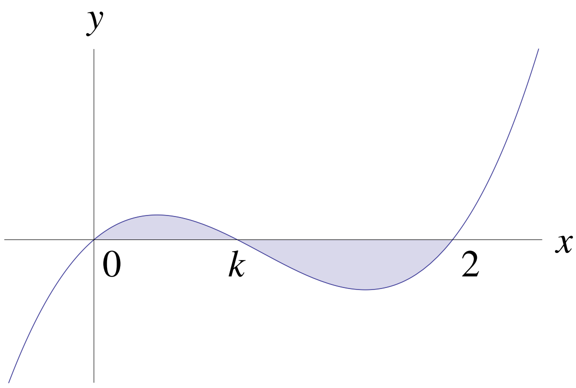
Turn over

3.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\\ \textbf{MATHEMATICS \& COMPUTER SCIENCE}\end{array}\right\}$ **ONLY.**

Computer Science and *Computer Science & Philosophy* **applicants should turn to page 14.**

Let $0 < k < 2$. Below is sketched a graph of $y = f_k(x)$ where $f_k(x) = x(x-k)(x-2)$. Let $A(k)$ denote the area of the shaded region.



(i) Without evaluating them, write down an expression for $A(k)$ in terms of two integrals.

(ii) Explain why $A(k)$ is a polynomial in k of degree 4 or less. [You are not required to calculate $A(k)$ explicitly.]

(iii) Verify that $f_k(1+t) = -f_{2-k}(1-t)$ for any t.

(iv) How can the graph of $y = f_k(x)$ be transformed to the graph of $y = f_{2-k}(x)$?

Deduce that $A(k) = A(2-k)$.

(v) Explain why there are constants a, b, c such that

$$A(k) = a(k-1)^4 + b(k-1)^2 + c.$$

[You are not required to calculate a, b, c explicitly.]

Turn over

4.

For **APPLICANTS IN** $\left\{ \begin{array}{l} \textbf{MATHEMATICS} \\ \textbf{MATHEMATICS \& STATISTICS} \\ \textbf{MATHEMATICS \& PHILOSOPHY} \end{array} \right\}$ **ONLY.**

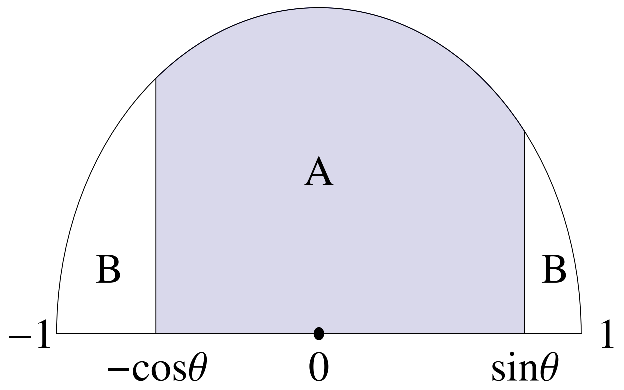
Mathematics & Computer Science, *Computer Science* and *Computer Science & Philosophy* **applicants should turn to page 14.**

(i) Let $a > 0$. On the axes opposite, sketch the graph of

$$y = \frac{a+x}{a-x} \qquad \text{for} \quad -a < x < a.$$

(ii) Let $0 < \theta < \pi/2$. In the diagram below is the half-disc given by $x^2 + y^2 \leqslant 1$ and $y \geqslant 0$. The shaded region A consists of those points with $-\cos\theta \leqslant x \leqslant \sin\theta$. The region B is the remainder of the half-disc.

Find the area of A.

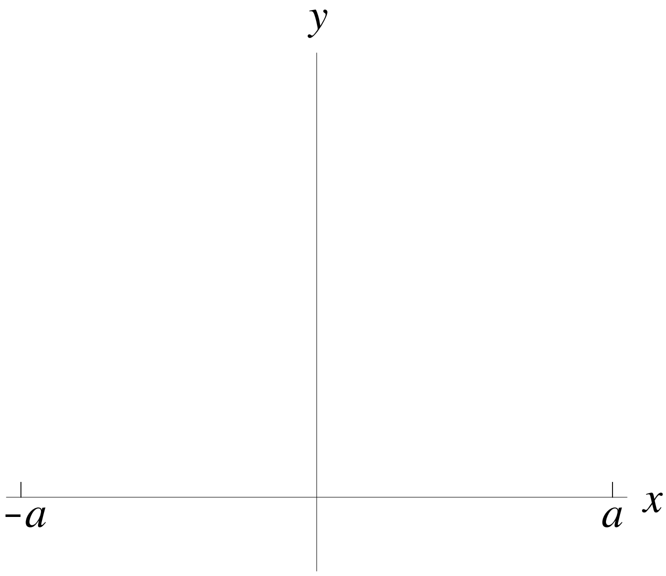


(iii) Assuming *only* that $\sin^2\theta + \cos^2\theta = 1$, show that $\sin\theta\cos\theta \leqslant 1/2$.

(iv) What is the largest that the ratio

$$\frac{\text{area of } A}{\text{area of } B}$$

can be, as θ varies?



Turn over

5. For **ALL APPLICANTS.**

We define the *digit sum* of a non-negative integer to be the sum of its digits. For example, the digit sum of 123 is $1 + 2 + 3 = 6$.

(i) How many positive integers less than 100 have digit sum equal to 8?

Let n be a positive integer with $n < 10$.

(ii) How many positive integers less than 100 have digit sum equal to n?

(iii) How many positive integers less than 1000 have digit sum equal to n?

(iv) How many positive integers between 500 and 999 have digit sum equal to 8?

(v) How many positive integers less than 1000 have digit sum equal to 8, and one digit at least 5?

(vi) What is the total of the digit sums of the integers from 0 to 999 inclusive?

Turn over

6.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{COMPUTER SCIENCE}\\ \textbf{MATHEMATICS \& COMPUTER SCIENCE}\\ \textbf{COMPUTER SCIENCE \& PHILOSOPHY}\end{array}\right\}$ **ONLY.**

Alice, Bob and Charlie are well-known expert logicians; they always tell the truth.

In each of the scenarios below, Charlie writes a whole number on Alice and Bob's foreheads. The difference between the two numbers is one: either Alice's number is one larger than Bob's, or Bob's number is one larger than Alice's. Each of Alice and Bob can see the number on the other's forehead, but can't see their own number.

(i) Charlie writes a number on Alice and Bob's foreheads, and says "Each of your numbers is at least 1. The difference between the numbers is 1."

Alice then says "I know my number."

Explain why Alice's number must be 2. What is Bob's number?

(ii) Charlie now writes new numbers on their foreheads, and says "Each of your numbers is between 1 and 10 inclusive. The difference between the numbers is 1. Alice's number is a prime." (A prime number is a number greater than 1 that is divisible only by 1 and itself.)

Alice then says "I don't know my number."

Bob then says "I don't know my number."

What is Alice's number? Explain your answer.

(iii) Charlie now writes new numbers on their foreheads, and says "Each of your numbers is between 1 and 10 inclusive. The difference between the numbers is 1."

Alice then says "I don't know my number. Is my number a square number?"

Charlie then says "If I told you that, you would know your number."

Bob then says "I don't know my number."

What is Alice's number? Explain your answer.

Turn over

7.

For **APPLICANTS IN** $\left\{ \begin{array}{l} \textbf{COMPUTER SCIENCE} \\ \textbf{COMPUTER SCIENCE \& PHILOSOPHY} \end{array} \right\}$ **ONLY.**

AB-words are "words" formed from the letters **A** and **B** according to certain rules. The rules are applied starting with the empty word, containing no letters. The basic rules are:

(1) If the current word is x, then it can be replaced with the word that starts with **A**, followed by x and ending with **B**, written $\mathbf{A}x\mathbf{B}$.

(2) If the current word ends with **B**, the final **B** can be removed.

(i) Show how the word **AAAB** can be produced.

(ii) Describe precisely all the words that can be produced with these two rules. Justify your answer. You might like to write $\mathbf{A}^i$ for the word containing just i consecutive copies of **A**, and similarly for **B**; for example $\mathbf{A}^3\mathbf{B}^2 = \mathbf{AAABB}$.

We now add a third rule:

(3) Reverse the word, and replace every **A** by **B**, and every **B** by **A**.

For example, applying this rule to **AAAB** would give **ABBB**.

(iii) Describe precisely all the words that can be produced with these three rules. Justify your answer.

Finally, we add a fourth rule:

(4) Reverse the word.

(iv) Show that every word consisting of **A**s and **B**s can be formed using these four rules. Hint: show how, if we have produced the word w, we can produce (a) the word $\mathbf{A}w$, and (b) the word $\mathbf{B}w$; hence deduce the result.

End of last question

This page has been intentionally left blank

This page has been intentionally left blank

This page has been intentionally left blank