OXFORD UNIVERSITY

MATHEMATICS ADMISSIONS TEST

Wednesday 2 November 2011

Time Allowed: 2½ hours

For candidates applying for Mathematics, Computer Science or one of their joint degrees.

Write your name, test centre (where you are sitting the test), Oxford college (to which you applied or were assigned) and your proposed degree course in BLOCK CAPITALS below.

UCAS PERSONAL ID:

NAME:

OXFORD COLLEGE (if known):

DEGREE COURSE:

DATE OF BIRTH:

This paper contains 7 questions of which you should attempt 5. There are directions throughout the paper as to which questions are appropriate for your course.

- **Mathematics, Maths & Philosophy, Maths & Statistics applicants should attempt 1,2,3,4,5.**
- **Mathematics & Computer Science applicants should attempt 1,2,3,5,6.**
- **Computer Science, Computer Science & Philosophy applicants should attempt 1,2,5,6,7.**

Further credit cannot be obtained by attempting extra questions.

Question 1 is a multiple choice question with ten parts. Marks are given solely for correct answers but any rough working should be shown in the space between parts. Answer Question 1 on the grid on Page 2. Each part is worth 4 marks.

Answers to questions 2-7 should be written in the space provided, continuing on to the blank pages at the end of this booklet if necessary. Each of Questions 2-7 is worth 15 marks.

For Test Supervisors Use Only:

[ ] Tick here if special arrangements were made for the test. Please either include details below or securely attach to the test script a letter with the details.

Signature of Invigilator: ______________________________

FOR OFFICE USE ONLY:

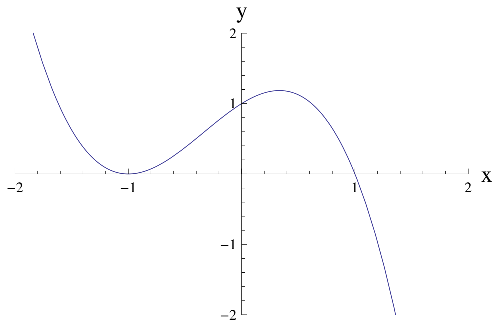
Q1	Q2	Q3	Q4	Q5	Q6	Q7

1. For **ALL APPLICANTS**.

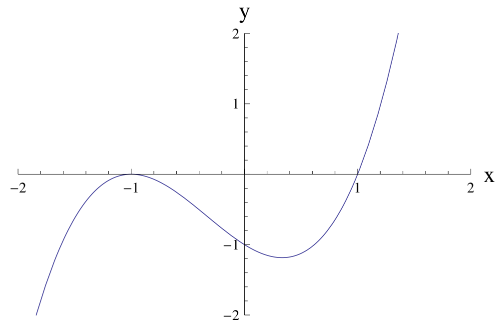
For each part of the question on pages 3–7 you will be given four possible answers, just one of which is correct. Indicate for each part **A**–**J** which answer (a), (b), (c), or (d) you think is correct with a tick (✓) in the corresponding column in the table below. *Please show any rough working in the space provided between the parts.*

	(a)	(b)	(c)	(d)
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

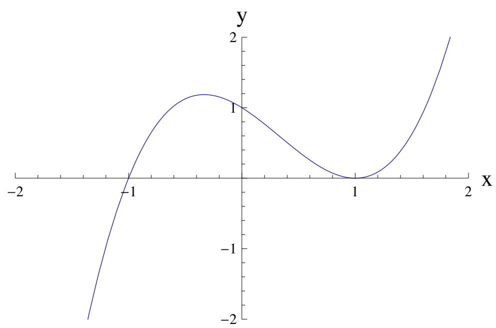
A. A sketch of the graph $y = x^3 - x^2 - x + 1$ appears on which of the following axes?



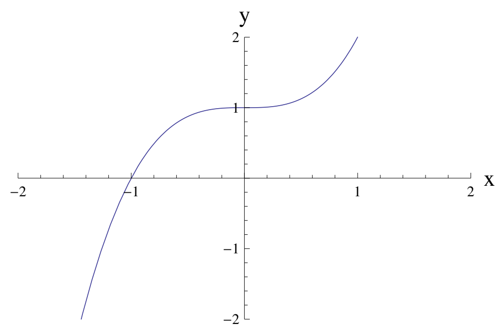
(a)



(b)



(c)



(d)

B. A rectangle has perimeter P and area A. The values P and A must satisfy

(a) $P^3 > A$, (b) $A^2 > 2P + 1$, (c) $P^2 \geqslant 16A$, (d) $PA \geqslant A + P$.

Turn over

C. The sequence x_n is given by the formula

$$x_n = n^3 - 9n^2 + 631.$$

The largest value of n for which $x_n > x_{n+1}$ is

(a) 5, (b) 7, (c) 11, (d) 17.

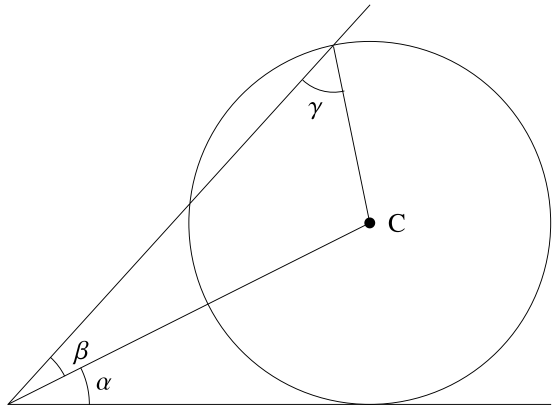
D. The fraction of the interval $0 \leqslant x \leqslant 2\pi$, for which one (or both) of the inequalities

$$\sin x \geqslant \frac{1}{2}, \qquad \sin 2x \geqslant \frac{1}{2}$$

is true, equals

(a) $\frac{1}{3}$, (b) $\frac{13}{24}$, (c) $\frac{7}{12}$, (d) $\frac{5}{8}$.

E. The circle in the diagram has centre C. Three angles α, β, γ are also indicated.



The angles α, β, γ are related by the equation:

(a) $\cos\alpha = \sin(\beta + \gamma)$;
(b) $\sin\beta = \sin\alpha \sin\gamma$;
(c) $\sin\beta\,(1 - \cos\alpha) = \sin\gamma$;
(d) $\sin(\alpha + \beta) = \cos\gamma \sin\alpha$.

F. Given θ in the range $0 \leqslant \theta < \pi$, the equation

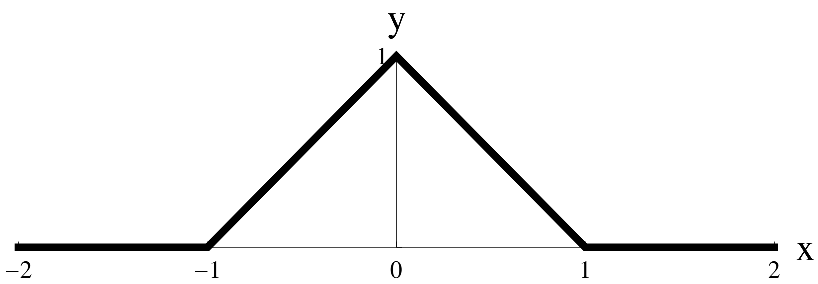
$$x^2 + y^2 + 4x\cos\theta + 8y\sin\theta + 10 = 0$$

represents a circle for

(a) $0 < \theta < \frac{\pi}{3}$, (b) $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$, (c) $0 < \theta < \frac{\pi}{2}$, (d) all values of θ.

Turn over

G. A graph of the function $y = f(x)$ is sketched on the axes below:



The value of $\int_{-1}^{1} f(x^2 - 1)\, \mathrm{d}x$ equals

$$\text{(a)}\quad \frac{1}{4}, \qquad \text{(b)}\quad \frac{1}{3}, \qquad \text{(c)}\quad \frac{3}{5}, \qquad \text{(d)}\quad \frac{2}{3}.$$

H. The number of *positive* values x which satisfy the equation

$$x = 8^{\log_2 x} - 9^{\log_3 x} - 4^{\log_2 x} + \log_{0.5} 0.25$$

is

(a) 0, (b) 1, (c) 2, (d) 3.

I. In the range $0 \leqslant x < 2\pi$ the equation

$$\sin^8 x + \cos^6 x = 1$$

has

(a) 3 solutions , (b) 4 solutions, (c) 6 solutions, (d) 8 solutions.

J. The function $f(n)$ is defined for positive integers n according to the rules

$$f(1) = 1, \qquad f(2n) = f(n), \qquad f(2n+1) = (f(n))^2 - 2.$$

The value of $f(1) + f(2) + f(3) + \cdots + f(100)$ is

(a) -86, (b) -31, (c) 23, (d) 58.

Turn over

2. For **ALL APPLICANTS.**

Suppose that x satisfies the equation

$$x^3 = 2x + 1. \qquad (*)$$

(i) Show that

$$x^4 = x + 2x^2 \qquad \text{and} \qquad x^5 = 2 + 4x + x^2.$$

(ii) For every integer $k \geqslant 0$, we can uniquely write

$$x^k = A_k + B_k x + C_k x^2$$

where A_k, B_k, C_k are integers. So, in part (i), it was shown that

$$A_4 = 0,\ B_4 = 1, C_4 = 2 \qquad \text{and} \qquad A_5 = 2,\ B_5 = 4,\ C_5 = 1.$$

Show that

$$A_{k+1} = C_k, \qquad B_{k+1} = A_k + 2C_k, \qquad C_{k+1} = B_k.$$

(iii) Let

$$D_k = A_k + C_k - B_k.$$

Show that $D_{k+1} = -D_k$ and hence that

$$A_k + C_k = B_k + (-1)^k.$$

(iv) Let $F_k = A_{k+1} + C_{k+1}$. Show that

$$F_k + F_{k+1} = F_{k+2}.$$

Turn over

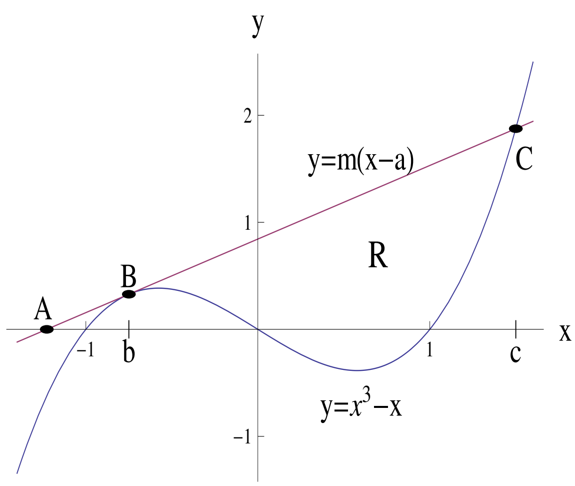
3.

For **APPLICANTS IN**
MATHEMATICS
MATHEMATICS & STATISTICS
MATHEMATICS & PHILOSOPHY
MATHEMATICS & COMPUTER SCIENCE
ONLY.

Computer Science and *Computer Science & Philosophy* **applicants should turn to page 14.**

The graphs of $y = x^3 - x$ and $y = m(x - a)$ are drawn on the axes below. Here $m > 0$ and $a \leqslant -1$.

The line $y = m(x - a)$ meets the x-axis at $A = (a, 0)$, *touches* the cubic $y = x^3 - x$ at B and intersects again with the cubic at C. The x-coordinates of B and C are respectively b and c.



(i) Use the fact that the line and cubic *touch* when $x = b$, to show that $m = 3b^2 - 1$.

(ii) Show further that

$$a = \frac{2b^3}{3b^2 - 1}.$$

(iii) If $a = -10^6$, what is the approximate value of b?

(iv) Using the fact that

$$x^3 - x - m(x - a) = (x - b)^2 (x - c)$$

(which you need not prove), show that $c = -2b$.

(v) R is the finite region bounded above by the line $y = m(x - a)$ and bounded below by the cubic $y = x^3 - x$. For what value of a is the area of R largest?

Show that the largest possible area of R is $\frac{27}{4}$.

Turn over

4.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\end{array}\right\}$ **ONLY.**

Mathematics & Computer Science, *Computer Science* and *Computer Science & Philosophy* **applicants should turn to page 14.**

Let Q denote the quarter-disc of points (x, y) such that $x \geqslant 0$, $y \geqslant 0$ and $x^2 + y^2 \leqslant 1$ as drawn in Figures A and B below.

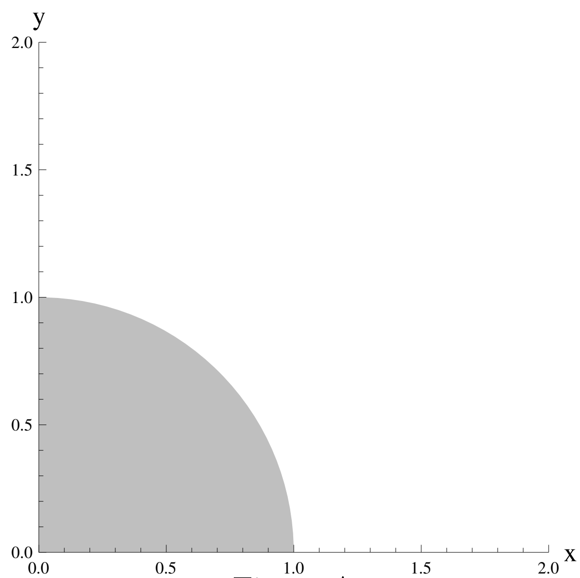


Figure A

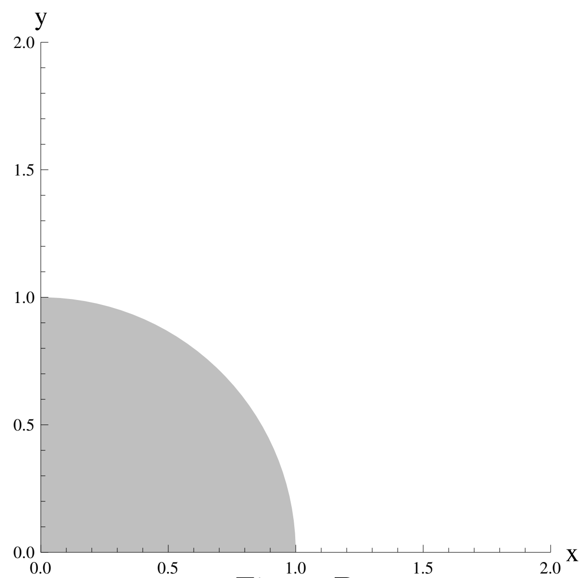


Figure B

(i) On the axes in Figure A, sketch the graphs of

$$x + y = \frac{1}{2}, \qquad x + y = 1, \qquad x + y = \frac{3}{2}.$$

What is the largest value of $x + y$ achieved at points (x, y) in Q? Justify your answer.

(ii) On the axes in Figure B, sketch the graphs of

$$xy = \frac{1}{4}, \qquad xy = 1, \qquad xy = 2.$$

What is the largest value of $x^2 + y^2 + 4xy$ achieved at points (x, y) in Q?

What is the largest value of $x^2 + y^2 - 6xy$ achieved at points (x, y) in Q?

(iii) Describe the curve

$$x^2 + y^2 - 4x - 2y = k$$

where $k > -5$.

What is the *smallest* value of $x^2 + y^2 - 4x - 2y$ achieved at points (x, y) in Q?

Turn over

5. For **ALL APPLICANTS.**

An $n \times n$ *grid* consists of squares arranged in n rows and n columns; for example, a chessboard is an 8×8 grid. Let us call a *semi-grid* of size n the lower left part of an $n \times n$ grid – that is, the squares located on or below the grid's diagonal. For example, Figure C shows an example of a semi-grid of size 4.

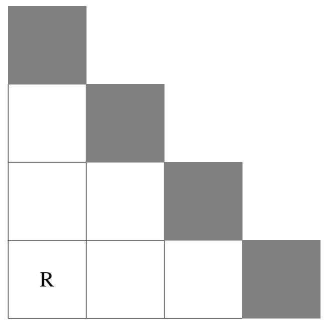


Figure C

Let us suppose that a robot is located in the lower-left corner of the grid. The robot can move only up or right, and its goal is to reach one of the *goal squares*, which are all located on the semi-grid's diagonal. In the example shown in Figure C, the robot is initially located in the square denoted with R, and the goal squares are shown in grey. Let us call a *solution* a sequence of the robot's moves that leads the robot from the initial location to some goal square.

(i) Write down all 8 solutions for a robot on a semi-grid of size 4.

(ii) Devise a concise way of representing the possible journeys of the robot in a semi-grid of size n. In your notation, which of the journeys are solutions?

(iii) Write down a formula for the number of possible solutions in a semi-grid of size n. Explain why your formula is correct.

Now let us change the problem slightly and redefine a goal square as any square that can be described as follows:

- the lower-left square is not a goal square;
- each square that is located immediately above or immediately to the right of a non-goal square is a goal square; and
- each square that is located immediately above or immediately to the right of a goal square is a non-goal square.

Furthermore, let us assume that, upon reaching a goal square, the robot may decide to stop or to continue moving (provided that there are more allowed moves).

(iv) With these modifications in place, write down all the solutions in a semi-grid of size 4, and all the solutions in a semi-grid of size 5.

(v) How many solutions are there now in a semi-grid of size n, where n is a positive integer? You may wish to consider separately the cases where n is even or odd.

Turn over

6.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{COMPUTER SCIENCE} \\ \textbf{MATHEMATICS \& COMPUTER SCIENCE} \\ \textbf{COMPUTER SCIENCE \& PHILOSOPHY}\end{array}\right\}$ **ONLY.**

Alice, Bob, Charlie and Diane are playing together when one of them breaks a precious vase. They all know who broke the vase. When questioned they make the following statements:

Alice:	It was Bob.
Bob:	It was Diane.
Charlie:	It was not me.
Diane:	What Bob says is wrong.

Each statement is either true or false.

(i) Explain why at least one of the four must be lying.

(ii) Explain why at least one of them must be telling the truth.

(iii) Let us suppose that exactly one of the four is lying, so the other three are telling the truth. Who is lying? Who did break the vase? Explain your answer.

(iv) Let us now suppose that exactly one of the four is telling the truth, so the other three are lying. Who is telling the truth? Who did break the vase? Explain your answer.

(v) Let us now suppose that two of the statements are true and two are false. List the people who might now have broken the vase. Justify your answers.

(vi) Hence show that if we don't know how many of the four statements are true, then any one of the four could have broken the vase.

Turn over

7.

For **APPLICANTS IN** $\left\{ \begin{array}{l} \textbf{COMPUTER SCIENCE} \\ \textbf{COMPUTER SCIENCE \& PHILOSOPHY} \end{array} \right\}$ **ONLY.**

Alice and Bob have a large bag of coins which they use to play a game called HT-2. In this game, Alice and Bob take turns placing one coin at a time on the table, each to the right of the previous one; thus they build a row of coins that grows to the right. Alice always places the first coin. Each coin is placed head-up (H) or tail-up (T), and cannot be flipped or moved once it has been placed.

A player loses the game if he or she places a coin that results in two adjacent coins having the same pattern of heads and tails as another adjacent pair somewhere in the row (reading a pattern from left to right). For example, Bob has lost this game by producing a second instance of HT (where a and b denote coins placed by Alice and Bob respectively):

$$\begin{array}{cccccc} a & b & a & b & a & b \\ H & H & T & T & H & T \end{array}$$

and Alice has lost this game by producing a second instance of TT (overlapping pairs can count as repeats):

$$\begin{array}{ccccc} a & b & a & b & a \\ T & H & T & T & T \end{array}$$

(i) What is the smallest number of coins that might be placed in a game of HT-2 (including the final coin that causes a player to lose)? What is the largest number? Justify each answer.

(ii) Bob can always win a game of HT-2 by adopting a particular strategy. Describe the strategy.

For any positive integer n, there is a game HT-n with the same rules as HT-2, except that the game is lost by a player who creates an unbroken sequence of n heads and tails that appears elsewhere in the row. For example, Bob has lost this game of HT-3 by producing a second instance of THT:

$$\begin{array}{cccccccc} a & b & a & b & a & b & a & b \\ H & H & T & T & H & T & H & T \end{array}$$

(iii) Suppose n is odd, and Bob chooses to play by always duplicating Alice's previous play (and Alice knows that this is Bob's strategy). Show that Alice can always win.

In these games, a maximum time of one minute is allowed for each turn.

(iv) Can we be certain that a game of HT-6 will be finished within two hours? Justify your answer.

End of last question

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